



TITLE:

Partial Sums of Certain Analytic Functions (Study on Applications for Fractional Calculus Operators in Univalent Function Theory)

AUTHOR(S):

Owa, Shigeyoshi; Srivastava, H.M.; Saito, Nobuyuki

CITATION:

Owa, Shigeyoshi ...[et al]. Partial Sums of Certain Analytic Functions (Study on Applications for Fractional Calculus Operators in Univalent Function Theory). 数理解析研究所講究録 2004, 1363: 96-114

ISSUE DATE:

2004-04

URL:

<http://hdl.handle.net/2433/25310>

RIGHT:

Partial Sums of Certain Analytic Functions

Shigeyoshi Owa, H.M.Srivastava and Nobuyuki Saito

Abstract

It is well-known that Koebe function $f(z) = \frac{z}{(1-z)^2}$ is the extremal function for the class S^* of starlike functions in the open unit disk \mathbb{U} , and that the function $g(z) = \frac{z}{1-z}$ is the extremal function for the class \mathcal{K} of convex functions in the open unit disk \mathbb{U} . But the partial sum $f_n(z)$ of $f(z)$ is not starlike in \mathbb{U} , and the partial sum $g_n(z)$ of $g(z)$ is not convex in \mathbb{U} . The object of the present paper is to discuss for starlikeness and convexity of the partial sums $f_n(z)$ and $g_n(z)$.

1 Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by \mathcal{S} the subclass of \mathcal{A} consisting of all univalent functions $f(z)$ in \mathbb{U} . Let $S^*(\alpha)$ be the subclass of \mathcal{A} consisting of all functions $f(z)$ which satisfy

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ in $S^*(\alpha)$ is said to be starlike of order α in \mathbb{U} . Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of \mathcal{A} consisting of all functions $f(z)$ which satisfy

2000 Mathematics Subject Classification: Primary 30C45.

Key Words and Phrases: Univalent function, starlike function, convex function, partial sum.

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha (0 \leq \alpha < 1)$. A function $f(z)$ belonging to $\mathcal{K}(\alpha)$ is said to be convex of order α in \mathbb{U} .

By the definitions for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, we note that $f(z) \in \mathcal{S}^*(\alpha)$ if and only if $zf'(z) \in \mathcal{K}(\alpha)$ and denote by $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$. It is well-known that Koebe function $f(z)$ given by

$$(1.4) \quad f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} kz^k$$

is the extremal function for the class \mathcal{S}^* , and that the function

$$(1.5) \quad g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$$

is the extremal function for the class \mathcal{K} .

For a function $f(z)$ given by (1.1), we introduce the partial sum of $f(z)$ by

$$(1.6) \quad f_n(z) = z + \sum_{k=2}^n a_k z^k.$$

For the partial sums $f_n(z)$ of $f(z) \in \mathcal{S}^*$, Szegő [4] showed the following result.

Theorem 1.1. *If $f(z) \in \mathcal{S}^*$, then $f_n(z) \in \mathcal{S}^*$ for $|z| < \frac{1}{4}$, and $f_n(z) \in \mathcal{K}$ for $|z| < \frac{1}{8}$.*

Further, Padmanabhan [3] proved the following theorem.

Theorem 1.2. *If $f(z)$ is 2-valently starlike in \mathbb{U} , then $f_n(z)$ is 2-valently starlike for $|z| < \frac{1}{6}$.*

Recently, Li and Owa [1] derived some interesting results for partial sums of the Libera integral operator which is defined by

$$L(f)(z) = \frac{2}{z} \int_0^z f(t) dt.$$

for $f(z) \in \mathcal{A}$, and Owa [2] considered partial sums for the extremal functions of the classes \mathcal{S}^* and \mathcal{K} .

Remark 1.1. If $f(z) \in \mathcal{S}$, then $f_n(z) \notin \mathcal{S}$ for $|a_n| \geq \frac{1}{n}$.

Proof. Note that

$$f'_n(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} = n a_n \left\{ z^{n-1} + \frac{(n-1)a_{n-1}}{n a_n} z^{n-2} + \dots + \frac{1}{n a_n} \right\} = 0$$

for $z = z_j$ ($j = 1, 2, 3, \dots, n-1$)

Therefore, we have

$$\left| \prod_{j=1}^{n-1} z_j \right| = \left| (-1)^{n-1} \frac{1}{n a_n} \right| \leq 1.$$

This shows that there exists a point $z_j \in \mathbb{U}$ such that $|z_j| < 1$. Thus we say that $f_n(z) \notin \mathcal{S}$ for $|a_n| \geq \frac{1}{n}$. \square

Noting that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$, we also have

- (i) $f_n(z) \notin \mathcal{S}^*$ for $f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \in \mathcal{S}^*$.
- (ii) $g_n(z) \notin \mathcal{K}$ for $g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{K}$.

2 Partial sums $f_3(z)$ and $g_3(z)$

For Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k$$

which is the extremal function for the class \mathcal{S}^* , we consider the partial sum $f_3(z) = z + 2z^2 + 3z^3$.

Theorem 2.1. The partial sum $f_3(z) = z + 2z^2 + 3z^3$ of Koebe function $f(z) = \frac{z}{(1-z)^2}$ satisfies

$$(2.1) \quad \operatorname{Re} \left(1 + \frac{z f_3''(z)}{f_3'(z)} \right) > \alpha(r) = 3 - \frac{2(1-2r)}{1-4r+9r^2} > 2 \left(1 - \frac{\sqrt{10}}{5} \right) = 0.7350 \dots$$

Where

$$0 \leq r < \frac{7-2\sqrt{10}}{9} = 0.0750 \dots$$

Proof. We consider α such that

$$(2.2) \quad \operatorname{Re} \left\{ 1 + \frac{zf_3''(z)}{f_3'(z)} \right\} = \operatorname{Re} \left\{ 3 - \frac{2(1+2z)}{1+4z+9z^2} \right\} > \alpha$$

for $0 \leq r < \frac{7-2\sqrt{10}}{9} = 0.0750\dots$.

It follows that

$$(2.3) \quad \operatorname{Re} \left\{ \frac{1+2z}{1+4z+9z^2} \right\} = \frac{1}{2} + \frac{(1-9r^2)(1+9r^2+4r\cos\theta)}{2(1-2r^2+81r^4+8r(1+9r^2)\cos\theta+36r^2\cos^2\theta)} \\ < \frac{3-\alpha}{2},$$

that is, that

$$(2.4) \quad \operatorname{Re} \left\{ \frac{(1-9r^2)(1+9r^2+4r\cos\theta)}{1-2r^2+81r^4+8r(1+9r^2)\cos\theta+36r^2\cos^2\theta} \right\} < 2-\alpha.$$

Let the function $g(t)$ be given by

$$(2.5) \quad g(t) = \frac{(1-9r^2)(1+9r^2+4rt)}{1-2r^2+81r^4+8r(1+9r^2)t+36r^2t^2} \quad (t = \cos\theta).$$

Then, we have

$$(2.6) \quad g'(t) \\ = \frac{-4r(1+3r)(1-3r)(1+38r^2-81r^3+162r^4+18r(1+4r+9r^2)t+36r^2t^2)}{(1-2r^2+81r^4+8r(1+9r^2)t+36r^2t^2)^2}.$$

Letting

$$(2.7) \quad h(t) = 1+38r^2-81r^3+162r^4+18r(1+4r+9r^2)t+36r^2t^2,$$

we see that

$$(i) \quad h(t) < 0 \implies g'(t) > 0,$$

$$(ii) \quad h(t) > 0 \implies g'(t) < 0,$$

and

$$(iii) \quad h(t) = 0 \text{ for } t = t_1, t = t_2 \quad (t_1 > t_2).$$

It is easy to see that $t_2 < -1$. Since

$$(2.8) \quad t_1 = \frac{-3(1+4r+9r^2) + \sqrt{5-72r+154r^2+972r^3+81r^4}}{12r},$$

our condition $0 \leq r < \frac{7-2\sqrt{10}}{9}$ implies that $t_1 \leq -1$, so that, $h(t) \geq 0$. Consequently, we conclude that

$$(2.9) \quad g(t) \leq g(-1) = \frac{1-9r^2}{1-4r+9r^2} < \frac{2\sqrt{10}}{5} \leq 2-\alpha,$$

that is,

$$\alpha = 2 - \frac{1-9r^2}{1-4r+9r^2} = 3 - \frac{2(1-2r)}{1-4r+9r^2}.$$

Thus, we have

$$\operatorname{Re} \left\{ 1 + \frac{zf_3''(z)}{f_3'(z)} \right\} > \alpha(r)$$

and

$$(2.10) \quad \alpha(r) = 3 - \frac{2(1-2r)}{1-4r+9r^2}$$

$$\text{for } 0 \leq r \leq \frac{7-2\sqrt{10}}{9}.$$

□

Next, for the function

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$$

which is the extremal function for the class \mathcal{K} , we consider the partial sum $g_3(z) = z + z^2 + z^3$.

Theorem 2.2. *The partial sum $g_3(z) = z + z^2 + z^3$ of the function $g(z) = \frac{z}{1-z}$ satisfies*

$$(2.11) \quad \operatorname{Re} \left(\frac{zg_3'(z)}{g_3(z)} \right) > \alpha(r) = 3 - \frac{2-r}{1-r+r^2} > \frac{4-\sqrt{5}}{2} = 0.9919\dots$$

Where

$$0 \leq r < \frac{7-3\sqrt{5}}{2} = 0.1458\dots$$

Proof. We consider α such that

$$(2.12) \quad \operatorname{Re} \left\{ \frac{zg'_3(z)}{g_3(z)} \right\} = \operatorname{Re} \left\{ 3 - \frac{2+z}{1+z+z^2} \right\} > \alpha$$

for $0 \leq r < \frac{7-3\sqrt{5}}{2} = 0.1458\dots$.

This implies that

$$(2.13) \quad \operatorname{Re} \left\{ \frac{2+z}{1+z+z^2} \right\} = 1 + \frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+4r^2\cos^2\theta+2r(1+r^2)\cos\theta} < 3-\alpha,$$

that is ,

$$(2.14) \quad \operatorname{Re} \left\{ \frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+4r^2\cos^2\theta+2r(1+r^2)\cos\theta} \right\} < 2-\alpha.$$

Let the function $g(t)$ be given by

$$(2.15) \quad g(t) = \frac{(1-r^2)(1+r^2+rt)}{1-r^2+r^4+4r^2t^2+2r(1+r^2)t} \quad (t = \cos\theta).$$

Then, we have

$$(2.16) \quad g'(t) = \frac{r(r+1)(r-1)(1+5r^2+r^4+4r^2t^2+8r(1+r^2)t)}{(1-r^2+r^4+4r^2t^2+2r(1+r^2)t)^2}.$$

Defining the function $h(t)$ by

$$(2.17) \quad h(t) = 1+5r^2+r^4+4r^2t^2+8r(1+r^2)t,$$

we see that

$$(i) \quad h(t) < 0 \implies g'(t) > 0,$$

$$(ii) \quad h(t) > 0 \implies g'(t) < 0,$$

and

$$(iii) \quad h(t) = 0 \text{ for } t = t_1, t = t_2 \quad (t_1 > t_2).$$

Note that $t_2 < -1$. Since

$$(2.18) \quad t_1 = \frac{-2(1+r^2) + \sqrt{3(1+r^2+r^4)}}{2r} < 0,$$

our condition $0 \leq r < \frac{7-3\sqrt{5}}{2}$ of the theorem implies that $t_1 \leq -1$, so that, $h(t) \geq 0$. Consequently, we conclude that

$$(2.19) \quad g(t) \leq g(-1) = \frac{1-r^2}{1-r+r^2} < \frac{4-\sqrt{5}}{2} \leq 2-\alpha,$$

that is,

$$\alpha = 2 - \frac{1-r^2}{1-r+r^2} = 3 - \frac{2-r}{1-r+r^2}.$$

Thus, we have

$$\operatorname{Re} \left\{ \frac{zg'_3(z)}{g_3(z)} \right\} > \alpha(r),$$

and

$$(2.20) \quad \alpha(r) = 3 - \frac{2-r}{1-r+r^2}$$

for $0 \leq r < \frac{7-3\sqrt{5}}{2} = 0.1458\dots$. □

Using the same method in the above, we also derive the following result.

Theorem 2.3. The partial sum $f_3(z) = z+2z^2+3z^3$ of Koebe function $f(z) = \frac{z}{(1-z)^2}$ satisfies

$$(2.21) \quad \operatorname{Re} \left\{ \frac{zf'_3(z)}{f_3(z)} \right\} > \alpha(r) = 3 - \frac{2(1-r)}{1-2r+3r^2} > \frac{3(89-16\sqrt{22})}{137} = 0.3055\dots$$

Where

$$0 \leq r < \frac{5-\sqrt{22}}{3} = 0.1031\dots$$

Theorem 2.4. The partial sum $g_3(z) = z+z^2+z^3$ of the function $g(z) = \frac{z}{1-z}$ satisfies

$$(2.22) \quad \operatorname{Re} \left\{ 1 + \frac{zg''_3(z)}{g'_3(z)} \right\} > \alpha(r) = 3 - \frac{2(1-r)}{1-2r+3r^2} > \frac{3(89-16\sqrt{22})}{137} = 0.3055\dots$$

Where

$$0 \leq r < \frac{5-\sqrt{22}}{3} = 0.1031\dots$$

3 Partial sums $f_4(z)$ and $g_4(z)$

For the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k$$

which is the extremal function for the class \mathcal{S}^* , we consider the partial sum

$$f_4(z) = z + 2z^2 + 3z^3 + 4z^4.$$

Theorem 3.1. *The partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ of Koebe function*

$f(z) = \frac{z}{(1-z)^2}$ satisfies

$$(3.1) \quad \operatorname{Re} \left\{ \frac{z f_4'(z)}{f_4(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}.$$

Where $0 \leq r \leq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545 \dots$$

Proof. For $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$, we have

$$\begin{aligned} (3.2) \quad \operatorname{Re} \left\{ \frac{z f_4'(z)}{f_4(z)} \right\} &= \operatorname{Re} \left\{ \frac{1 + 4z + 9z^2 + 16z^3}{1 + 2z + 3z^2 + 4z^3} \right\} \\ &= 4 - \operatorname{Re} \left\{ \frac{3 + 4z + 3z^2}{1 + 2z + 3z^2 + 4z^3} \right\} \\ &= 4 - \operatorname{Re} \left\{ \frac{3 + 4re^{i\theta} + 3r^2e^{i2\theta}}{1 + 2re^{i\theta} + 3r^2e^{i2\theta} + 4r^3e^{i3\theta}} \right\} \quad (z = re^{i\theta}). \end{aligned}$$

By using Mathematica, we know that the value of (3.2) takes its minimum value for $\theta = \pi$. This gives that

$$(3.3) \quad \operatorname{Re} \left\{ \frac{z f_4'(z)}{f_4(z)} \right\} \geq 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \quad (0 \leq r \leq r_0).$$

Let the function $h(r)$ be given by

$$h(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \quad (0 \leq r \leq r_0).$$

Since $0 = h(r_0) \leq h(r) \leq 1$
for

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545 \dots,$$

we have

$$(3.4) \quad \operatorname{Re} \left\{ \frac{zf'_4(z)}{f_4(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}$$

which completes the proof of the theorem. □

Next, we show

Theorem 3.2. *The partial sum $g_4(z) = z + z^2 + z^3 + z^4$ of the function $g(z) = \frac{z}{1-z}$ satisfies*

$$(3.5) \quad \operatorname{Re} \left\{ 1 + \frac{zg''_4(z)}{g'_4(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}.$$

Where $0 \leq r \leq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545 \dots$$

Proof. For $g_4(z) = z + z^2 + z^3 + z^4$, we have

$$\begin{aligned} (3.6) \quad \operatorname{Re} \left\{ 1 + \frac{zg''_4(z)}{g'_4(z)} \right\} &= \operatorname{Re} \left\{ 1 + \frac{2z + 6z^2 + 12z^3}{1 + 2z + 3z^2 + 4z^3} \right\} \\ &= 4 - \operatorname{Re} \left\{ \frac{3 + 4z + 3z^2}{1 + 2z + 3z^2 + 4z^3} \right\} \\ &= 4 - \operatorname{Re} \left\{ \frac{3 + 4re^{i\theta} + 3r^2e^{i2\theta}}{1 + 2re^{i\theta} + 3r^2e^{i2\theta} + 4r^3e^{i3\theta}} \right\} \quad (0 \leq r \leq r_0) \end{aligned}$$

Further, an application of Mathematica shows that

$$(3.7) \quad \operatorname{Re} \left\{ 1 + \frac{zg''_4(z)}{g'_4(z)} \right\} \geq 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \quad (0 \leq r \leq r_0).$$

Defining the function $h(r)$ by

$$h(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3} \quad (0 \leq r \leq r_0),$$

we see that

$$0 \leq h(r_0) \leq h(r) \leq 1$$

for

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545\dots,$$

that is, that

$$(3.8) \quad \operatorname{Re} \left\{ 1 + \frac{zg_4''(z)}{g_4'(z)} \right\} > \alpha(r) = 4 - \frac{3 - 4r + 3r^2}{1 - 2r + 3r^2 - 4r^3}$$

for $0 \leq r \leq r_0$. □

Using the same method in the above, we also derive

Theorem 3.3. The partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ of Koebe function $f(z) = \frac{z}{(1-z)^2}$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf_4''(z)}{f_4'(z)} \right\} > \alpha(r) = 4 - \frac{3 - 8r + 9r^2}{1 - 4r + 9r^2 - 16r^3}.$$

Where $0 \leq r \leq r_1 < 1$ and

$$r_1 = \frac{9}{64} + \frac{\sqrt[3]{4257 + 64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3(4257 + 64\sqrt{18681})}} = 0.1933\dots$$

Theorem 3.4. The partial sum $g_4(z) = z + z^2 + z^3 + z^4$ of the function $g(z) = \frac{z}{1-z}$ satisfies

$$\operatorname{Re} \left\{ \frac{zg_4'(z)}{g_4(z)} \right\} > \alpha(r) = 4 - \frac{3 - 2r + r^2}{(1-r)(1+r^2)}.$$

Where $0 \leq r \leq r_2 < 1$ and

$$r_2 = \frac{1}{4} + \frac{\sqrt[3]{5(9 + 4\sqrt{6})}}{4\sqrt[3]{9}} - \frac{\sqrt[3]{25}}{4\sqrt[3]{3(9 + 4\sqrt{6})}} = 0.6058\dots$$

4 Appendix

In this section, we try to describe the image domain of the disk by the partial sums for the theorems in Section 2 and Section 3.

Example 4.1. By Theorem 2.1, we take the partial sum $f_3(z) = z + 2z^2 + 3z^3$ for $|z| = r$ with

$$0 \leq r < \frac{7 - 2\sqrt{10}}{9} = 0.0750\dots$$

The image domain of $f_3(z)$ is shown in Fig.4.1.

```
<< Graphics`ComplexMap`
f3[z_] = z + 2 z^2 + 3 z^3
Pmf3 := PolarMap[f3, {0,  $\frac{7 - 2\sqrt{10}}{9}$ }, {0, 2  $\pi$ }]
Show[Pmf3]
```

$z + 2 z^2 + 3 z^3$

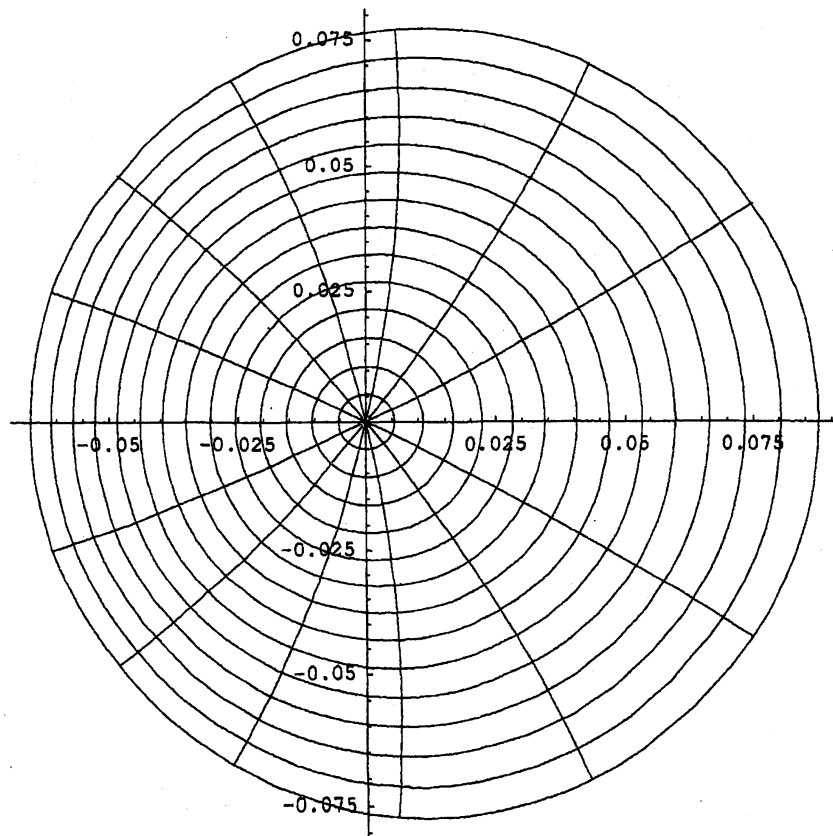


Fig.4.1

Example 4.2. By Theorem 2.2, we take the partial sum $g_3(z) = z + z^2 + z^3$ for $|z| = r$ with

$$0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\dots$$

The image domain of $g_3(z)$ is given by Fig.4.2.

```
<< Graphics`ComplexMap`
g3[z_] = z + z^2 + z^3
Pmg3; = PolarMap[g3, {0,  $\frac{7 - 3\sqrt{5}}{2}$ }, {0, 2\pi}];
Show[Pmg3]
```

$z + z^2 + z^3$

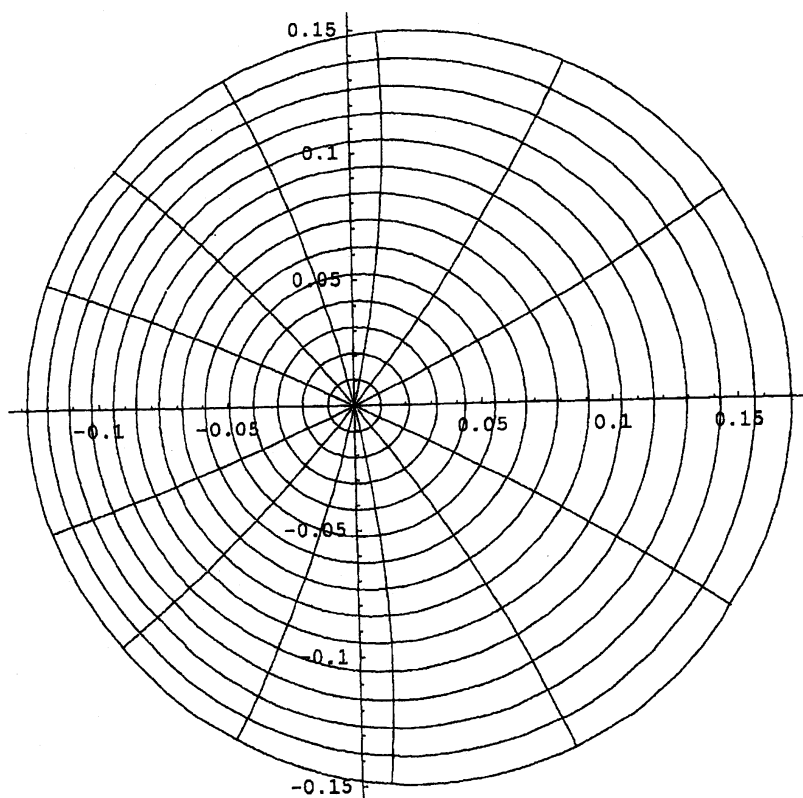


Fig.4.2

Example 4.3. By Theorem 2.3, we consider the partial sum $f_3(z) = z + 2z^2 + 3z^3$ for $|z| = r$ with

$$0 \leq r < \frac{5 - \sqrt{22}}{3} = 0.1031 \dots$$

We give the image domain of $f_3(z)$ in Fig.4.3.

```
<< Graphics`ComplexMap`
f3[z_] = z + 2 z^2 + 3 z^3
Pmf3 := PolarMap[f3, {0,  $\frac{5 - \sqrt{22}}{3}$ }, {0, 2 \pi}];
Show[Pmf3]
```

$$z + 2 z^2 + 3 z^3$$

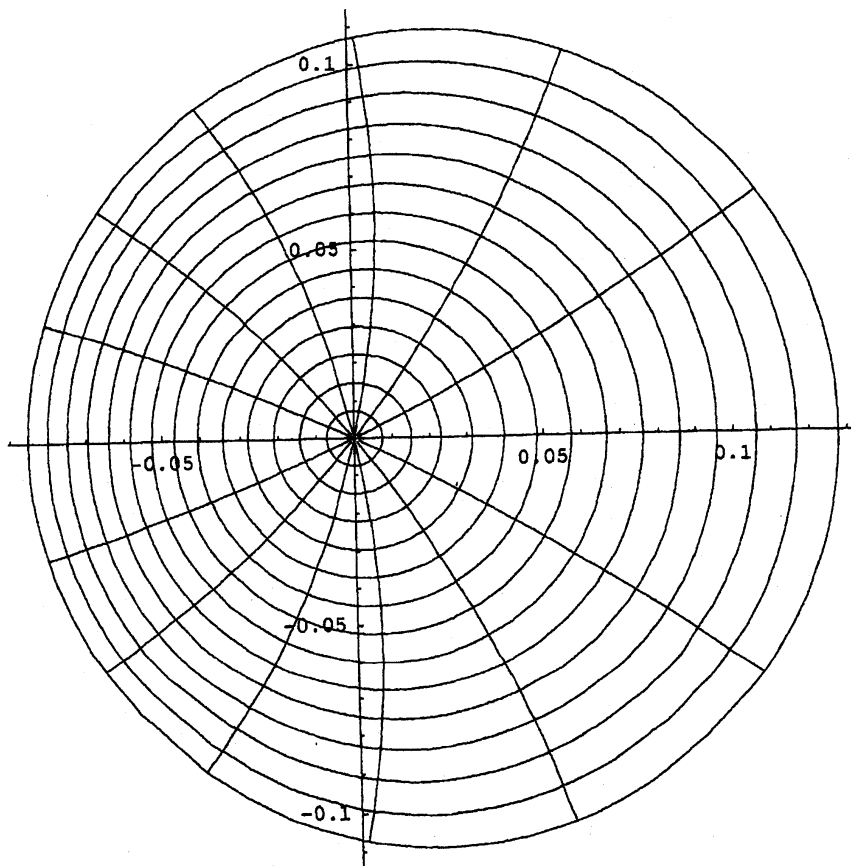


Fig.4.3

Example 4.4. By Theorem 2.4, we take the partial sum $g_3(z) = z + z^2 + z^3$ for $|z| = r$ with

$$0 \leq r < \frac{7 - 3\sqrt{5}}{2} = 0.1458\dots$$

The image domain of $g_3(z)$ is shown in Fig.4.4.

```
<< Graphics`ComplexMap`
g3[z_] = z + z^2 + z^3
Pmg3 := PolarMap[g3, {0,  $\frac{5 - \sqrt{22}}{3}$ }, {0, 2\pi}];
Show[Pmg3]
```

$z + z^2 + z^3$

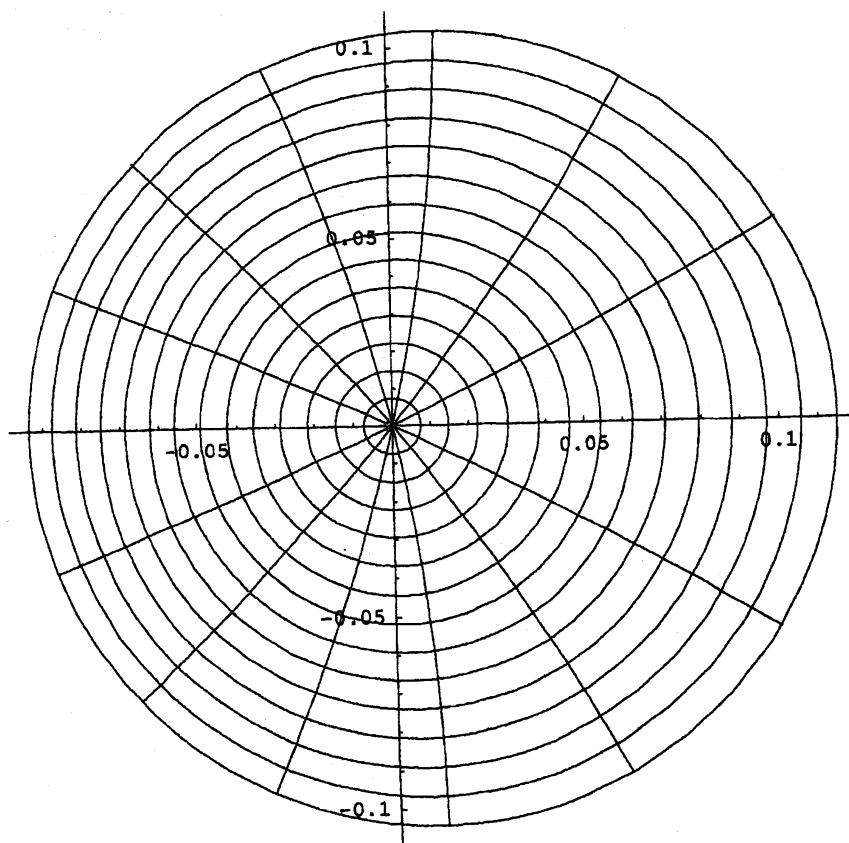


Fig.4.4

Example 4.5. By Theorem 3.1, we consider the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ for $|z| = r$ with $0 \leq r \leq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{531 + 16\sqrt{1695}}} = 0.3545 \dots$$

Then we show the image domain of $f_4(z)$ in Fig.4.5.

```
<< Graphics`ComplexMap`
f4[z_] = z + 2 z^2 + 3 z^3 + 4 z^4
Pmf4 := PolarMap[f4, {0, r0}, {0, 2 π}];

r0 = 3/16 + (531 + 16 Sqrt[1695])^(1/3)/(16 3^(2/3)) - 37/(16 3 (531 + 16 Sqrt[1695])^(1/3))

Show[Pmf4]

z + 2 z^2 + 3 z^3 + 4 z^4

3/16 + (531 + 16 Sqrt[1695])^(1/3)/(16 3^(2/3)) - 37/(16 3 (531 + 16 Sqrt[1695])^(1/3))
```

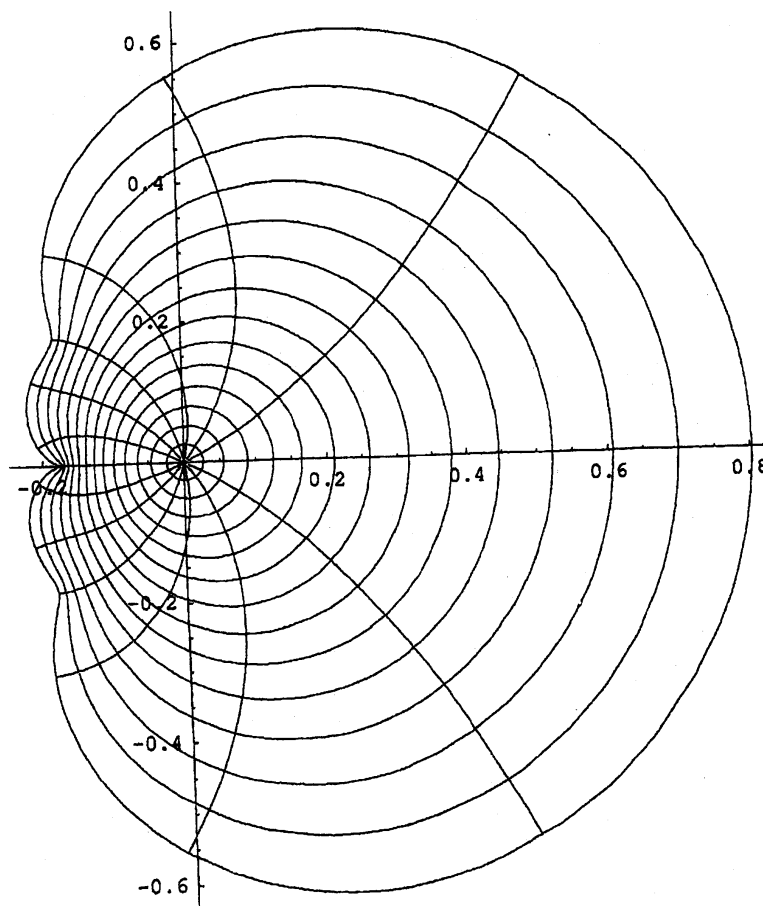


Fig.4.5

Example 4.6. By Theorem 3.2, we consider the partial sum $g_4(z) = z + z^2 + z^3 + z^4$ for $|z| = r$ with $0 \leq r \leq r_0 < 1$ and

$$r_0 = \frac{3}{16} + \frac{\sqrt[3]{531 + 16\sqrt{1695}}}{16\sqrt[3]{9}} - \frac{37}{16\sqrt[3]{3(531 + 16\sqrt{1695})}} = 0.3545\dots$$

Then we have the image domain of $g_4(z)$ by Fig.4.6.

```
<< Graphics`ComplexMap`
g4[z_] = z + z^2 + z^3 + z^4
Pmg4 := PolarMap[g4, {0, r0}, {0, 2 π}];

r0 = 3/16 + (531 + 16 Sqrt[1695])^(1/3)/(16 3^(2/3)) - 37/(16 (3 (531 + 16 Sqrt[1695]))^(1/3))

Show[Pmg4]
```

$$z + z^2 + z^3 + z^4$$

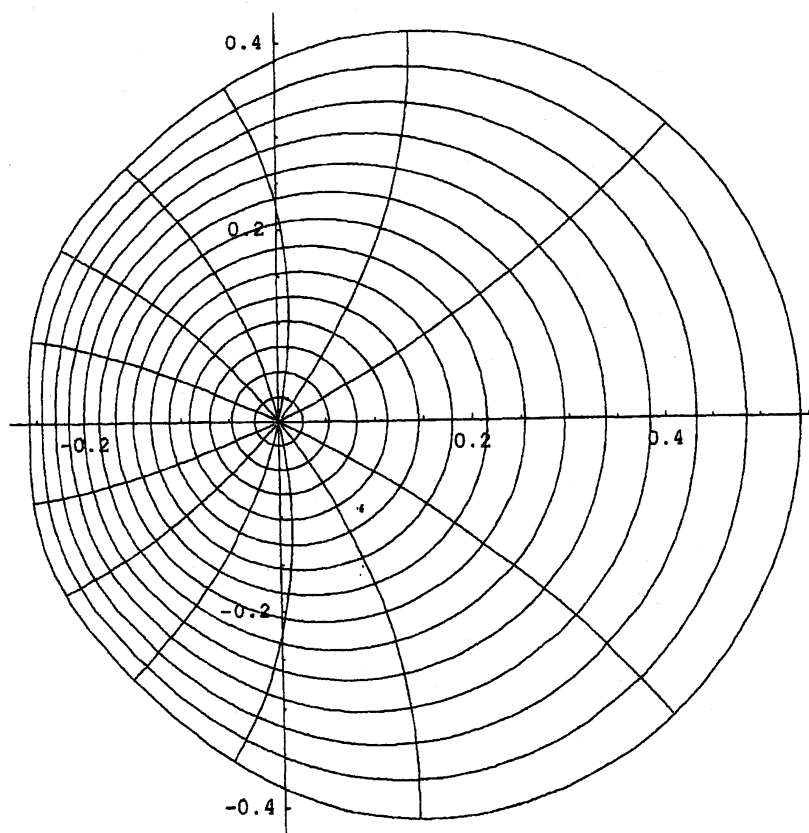
$$\frac{3}{16} + \frac{(531 + 16\sqrt{1695})^{1/3}}{16 \cdot 3^{2/3}} - \frac{37}{16 (3 (531 + 16\sqrt{1695}))^{1/3}}$$


Fig.4.6

Example 4.7. Considering the partial sum $f_4(z) = z + 2z^2 + 3z^3 + 4z^4$ for $|z| = r$ with $0 \leq r \leq r_1 < 1$ and

$$r_1 = \frac{9}{64} + \frac{\sqrt[3]{4257 + 64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3(4257 + 64\sqrt{18681})}} = 0.1933\dots,$$

in Theorem 3.3, we see the image domain of $f_4(z)$ in Fig.4.7.

```
<< Graphics`ComplexMap`
f4[z_] = z + 2 z^2 + 3 z^3 + 4 z^4
Pmf4 := PolarMap[f4, {0, r1}, {0, 2 π}];
```

$$r1 = \frac{9}{64} + \frac{\sqrt[3]{4257 + 64\sqrt{18681}}}{64\sqrt[3]{9}} - \frac{269}{64\sqrt[3]{3(4257 + 64\sqrt{18681})}}$$

```
Show[Pmf4]
```

$$z + 2 z^2 + 3 z^3 + 4 z^4$$

$$\frac{9}{64} + \frac{(4257 + 64\sqrt{18681})^{1/3}}{64 \cdot 3^{2/3}} - \frac{269}{64 (3 (4257 + 64\sqrt{18681}))^{1/3}}$$

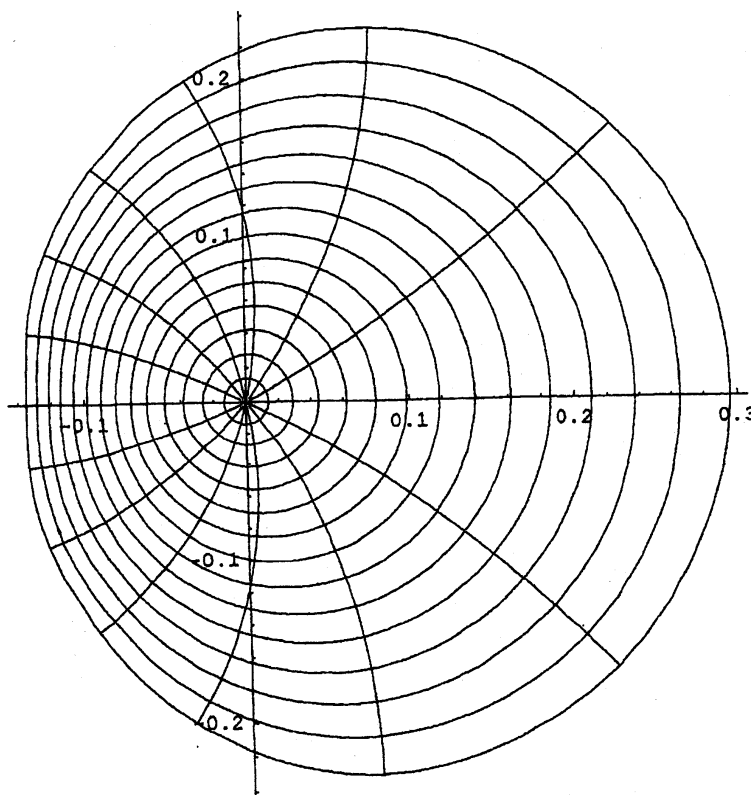


Fig.4.7

Example 4.8. Taking the partial sum $g_4(z) = z + z^2 + z^3 + z^4$ for $|z| = r$ with $0 \leq r \leq r_2 < 1$ and

$$r_2 = \frac{1}{4} + \frac{\sqrt[3]{5(9+4\sqrt{6})}}{4\sqrt[3]{9}} - \frac{\sqrt[3]{25}}{4\sqrt[3]{3(9+4\sqrt{6})}} = 0.6058\dots,$$

we have the image domain of $g_4(z)$ in Fig.4.8.

```
<< Graphics`ComplexMap`
g4[z_] = z + z^2 + z^3 + z^4
Pmg4 := PolarMap[g4, {0, r2}, {0, 2 π}];

r2 = 1/4 + (5^(1/3)*(9+4*sqrt(6))^(1/3))/(4*9^(1/3)) - (25^(1/3))/(4*(3*(9+4*sqrt(6)))^(1/3))

Show[Pmg4]
```

$$z + z^2 + z^3 + z^4$$

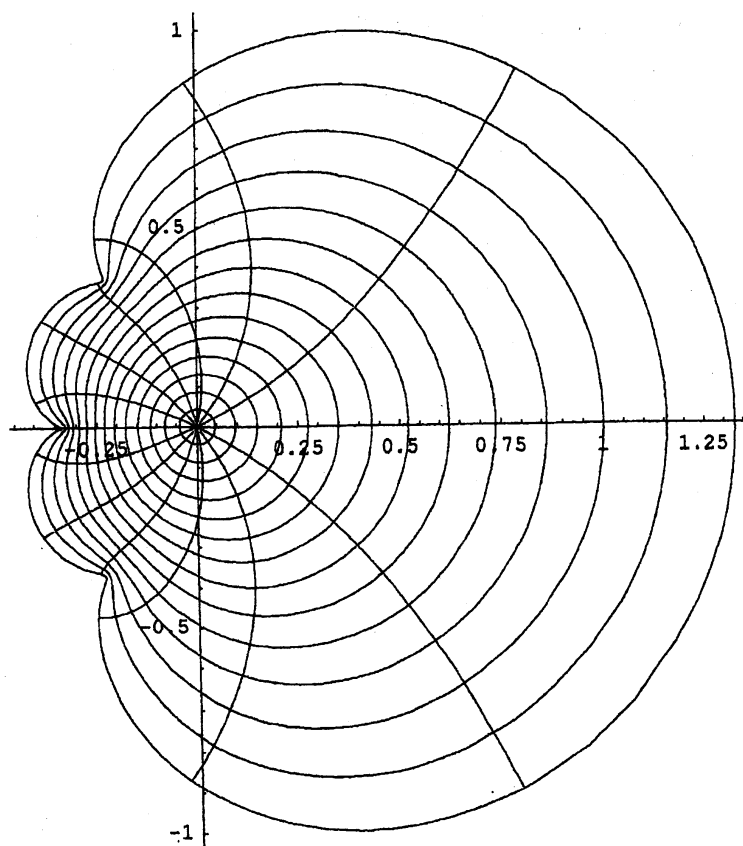
$$\frac{1}{4} - \frac{5^{2/3}}{4(3(9+4\sqrt{6}))^{1/3}} + \frac{(5(9+4\sqrt{6}))^{1/3}}{4 \cdot 3^{2/3}}$$


Fig.4.8

References

- [1] J.L.Li and S.Owa, *Partial sums of the Libera integral operator*, J. Math. Anal. Appl. **213**(1997), 444 – 454.
- [2] S.Owa, *Partial sums of certain analytic functions*, Internat. J. Math. Math. Sci. **25**(2001), 771 – 775.
- [3] K.S.Padmanabhan, *On the partial sums of certain analytic functions in the unit disc*, Ann. Poln. Math. **23**(1970/1971), 83 – 92.
- [4] G.Szegö, *Zur theorie der schlichten abbildungen*, Math. Ann. **100**(1928), 188 – 211.

S. Owa :

Department of Mathematics

Kinki University

Higashi-Osaka, Osaka 577-8502

Japan

E-mail: owa@math.kindai.ac.jp

H.M.Srivastava :

Department of Mathematics and Statistics

University of Victoria

Victoria, British Columbia V8W 3P4

Canada

E-mail: hmsri@uvvm.uvic.ca

N.Saito :

Department of Mathematics

Kinki University

Higashi-Osaka, Osaka 577-8502

Japan